Math 113 (Calculus II)
Exam 1 RED KEY

Part I: Multiple Choice *Mark the correct answer on the bubble sheet provided. Responses written on your exam will be ignored.*

1. What is the area between $f(x) = \cos(x)$ and $g(x) = \sin(2x) + 1$ for $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$?

   a) π
   b) 2π
   c) $\pi/2$
   d) $1 + \pi$
   e) $2 + \pi$
   f) $1 + 2\pi$

   Solution: The answer is e):

$$\int_{\pi/2}^{3\pi/2} \sin(2x) + 1 - \cos(x)\, dx = -\frac{1}{2}\cos(2x) + x - \sin(x)|_{\pi/2}^{3\pi/2}$$

$$= -\frac{1}{2}(-1) + \frac{3\pi}{2} - (-1) - [-\frac{1}{2}(-1) + \frac{\pi}{2} - (1)] = \pi + 2.$$

2. Find the volume of the solid obtained by revolving around the x-axis the region bounded by $y = x$, $x = 1$, and the x-axis.

   a) $\pi/3$
   b) $\pi/4$
   c) $\pi/5$
   d) $\pi/6$
   e) $\pi/8$
   f) $\pi/10$

   Solution: The answer is a):

$$\int_0^1 \pi(x)^2\, dx = \frac{1}{3}x^3|_0^1 = \frac{\pi}{3}$$

   Note: you can do this another way - Revolving $y = x$ about the x axis creates a cone. In this case, the radius is 1 and the height is 1. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$, so you can solve it geometrically, if you remember the formula.

3. Find the volume generated by rotating the region bounded by $y = x^2, y = 0, x = 1$ about the y-axis.

   a) $2\pi/3$
   b) $\pi/2$
   c) $\pi/3$
   d) 2π
   e) 1
   f) None of the above

   Solution: The answer is b):

$$\int_0^1 2\pi x \cdot x^2\, dx = 2\pi \int_0^1 x^3\, dx = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$$

4. A force of 3 lbs is required to hold a spring stretched 1/10 feet beyond its natural length. How much work (in ft-lbs) is done in stretching the spring from its natural length to 1/2 feet beyond its natural length?

a) 11/4
b) 13/4
c) 15/4
d) 17/4
e) 19/4
f) 21/4

Solution: The answer is c): Using Hooke's law: $F = kx$, we see that

$$3 = \frac{k}{10}, \text{ or } k = 30.$$

The work is calculated by integrating the force:

$$W = \int_0^{1/2} 30x\,dx = 15x^2|_0^{1/2} = \frac{15}{4}.$$

5. Find the average value of the function $f(x) = 4x\sin(x^2)$ on the interval $[0, \sqrt{\pi}]$.

a) 8
b) 4
c) $8/\pi$
d) $8/\sqrt{\pi}$
e) $4/\sqrt{\pi}$
f) $4/\pi$

Solution: The answer is e): The average value is given by

$$\frac{1}{\sqrt{\pi}-0}\int_0^{\sqrt{\pi}} 4x\sin(x^2)\,dx.$$

Using $u = x^2$ and $du = 2x\,dx$, the above integral becomes

$$\frac{1}{\sqrt{\pi}}\int_0^{\pi} 2\sin(u)\,du = -\frac{2}{\sqrt{\pi}}\cos(u)|_0^{\pi} = -\frac{2}{\sqrt{\pi}}(-1-1) = \frac{4}{\sqrt{\pi}}.$$

6. Evaluate $\int_0^1 (x+1)e^x\,dx$.

a) $e - 2$
b) e
c) $3e - 3$
d) $-e$
e) $2 - e$
f) none of the above

Solution: The answer is b): Use integration by parts. Let $u = x + 1$, and $dv = e^x\,dx$. Then, $du = dx$ and $v = e^x$.

$$\int_0^1 (x+1)e^x\,dx = [(x+1)e^x - \int e^x\,dx]_0^1 = xe^x|_0^1 = e.$$

7. Evaluate $\int_0^{\pi/2} \sin^3 x \cos^3 x dx$.

a) 1
b) 1/2
c) 1/4
d) 1/12
e) 1/24
f) 1/48

Solution: The answer is d):

$$\int_0^{\pi/2} \sin^3 x \cos^3 x dx = \int_0^{\pi/2} \sin^3 x \cos^2 x \cos x dx = \int_0^{\pi/2} \sin^3 x(1 - \sin^2 x) \cos x dx$$

Let $u = \sin x$. Then $du = \cos x\, dx$, and the above integral becomes

$$\int_0^1 u^3(1 - u^2)\, du = \int_0^1 u^3 - u^5\, du = (\frac{1}{4}u^4 - \frac{1}{6}u^6)|_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

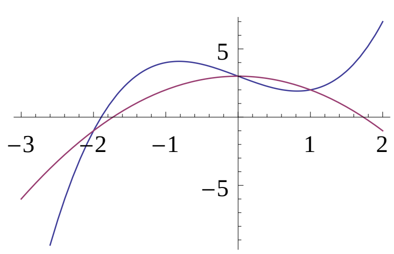
Part II: *In the following problems, show all work, and simplify your results.*

8. (10 points) Find the finite area bounded by $h(x) = x^3 - 2x + 3$ and $k(x) = 3 - x^2$.

Solution: First we find the points where the graphs of the two functions intersect.

$$\begin{aligned} h(x) &= k(x) \\ x^3 - 2x + 3 &= 3 - x^2 \\ x^3 + x^2 - 2x &= 0 \\ x(x^2 + x - 2) &= 0 \\ x(x + 2)(x - 1) &= 0. \end{aligned}$$

The curves intersect when $x = 0$, $x = 1$, and $x = -2$. Here is a sketch:



So, the area A is

$$\begin{aligned}
A &= \int_{-2}^{1} |h(x) - k(x)|\,dx = \int_{-2}^{0} [h(x) - k(x)]\,dx + \int_{0}^{1} [k(x) - h(x)]\,dx \\
&= \int_{-2}^{0} [(x^3 - 2x + 3) - (3 - x^2)]\,dx + \int_{0}^{1} [(3 - x^2) - (x^3 - 3x + 3)]\,dx \\
&= \int_{-2}^{0} (x^3 + x^2 - 2x)\,dx + \int_{0}^{1} (-x^3 - x^2 + 2x)\,dx \\
&= \left(\frac{x^4}{4} + \frac{x^3}{3} - x^2\right)\Big|_{-2}^{0} + \left(-\frac{x^4}{4} - \frac{x^3}{3} + x^2\right)\Big|_{0}^{1} \\
&= \left(\frac{0^4}{4} + \frac{0^3}{3} - 0^2\right) - \left(\frac{(-2)^4}{4} + \frac{(-2)^3}{3} - (-2)^2\right) + \left(-\frac{1^4}{4} - \frac{1^3}{3} + 1^2\right) - \left(-\frac{0^4}{4} - \frac{0^3}{3} + 0^2\right) \\
&= 0 - \left(4 - \frac{8}{3} - 4\right) + \left(-\frac{1}{4} - \frac{1}{3} + 1\right) - 0 = \frac{8}{3} + \left(-\frac{3}{12} - \frac{4}{12} + \frac{12}{12}\right) = \frac{8}{3} + \frac{5}{12} = \frac{32}{12} + \frac{5}{12} \\
&= \boxed{\frac{37}{12}}
\end{aligned}$$

9. (10 points) Find the volume of the solid whose base is the region bounded by $y = x^3$, $y = 1$, and the y-axis, and whose cross-sections perpendicular to the y-axis are equilateral triangles.

Solution: Since the cross section goes from the y axis to $y = x^3$, the cross section is $x = y^{1/3}$. Since the area of an equilateral triangle with side s is $\frac{\sqrt{3}}{4}s^2$, the the cross sectional area of this object is $A(y) = \frac{\sqrt{3}}{4}(y^{1/3})^2 = \frac{\sqrt{3}}{4}y^{2/3}$. Thus, the volume is

$$V = \int_0^1 \frac{\sqrt{3}}{4} y^{2/3}\,dy = \frac{\sqrt{3}}{4} \cdot \frac{3}{5} y^{5/3}|_0^1 = \frac{3\sqrt{3}}{20}.$$

10. (10 points) Use CYLINDRICAL SHELLS to find the volume generated by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

Solution:

$$V = \int_0^1 2\pi(2 - x)(x - x^2)\,dx = 2\pi \int_0^1 x^3 - 3x^2 + 2x\,dx = 2\pi(\frac{1}{4}x^4 - x^3 + x^2)|_0^1$$

$$= 2\pi(\frac{1}{4} - 1 + 1) = \frac{\pi}{2}$$

11. (10 points) A 20 foot chain weighs 30 lbs per foot and hangs vertically from a ceiling. Find the work done in lifting the lower end of the chain so that it's level with the upper end.

Solution: There are many equivalent ways to solve this problem. Here is one:

Notice, if you are lifting the chain to the top of the ceiling, if you have gone up x feet, then there is $20 - x$ feet of chain above you. The rest of the chain is doubled in half, however, and the rest of the chain is x feet. That means you are holding $x/2$ feet of chain. Hence, the force function is $F(x) = 30 \cdot x/2 = 15x$. The work is

$$W = \int_0^2 015x\,dx = \frac{15}{2}x^2|_0^{20} = \frac{15}{2} \cdot (20)^2 = \frac{15}{2} \cdot 400 = 3000 \text{ ft-lbs}.$$

12. (10 points) For $f(x) = 1 - 6x + 3x^2$, find the number(s) x such that $f(x)$ is equal to the average value of $f(x)$ on the interval $[0, 2]$.

Solution: The average value of f is

$$\frac{1}{2-0}\int_0^2 1 - 6x + 3x^2\,dx = \frac{1}{2}(x - 3x^2 + x^3)|_0^2 = \frac{1}{2}(2 - 3\cdot 4 + 8) = -1.$$

We need to find c so that $f(c) = -1$. Thus,

$$1 - 6c + 3c^2 = -1,$$

or

$$3c^2 - 6c + 2 = 0.$$

Applying the quadratic formula, we have $c = \frac{6\pm\sqrt{36-24}}{6}$, or $c = 1 \pm \frac{\sqrt{3}}{3}$. Note that both of these are in the interval in question.

13. (7 points) Evaluate $\int e^x \cos x\ dx$.

Solution: We use integration by parts: $u = e^x$, $du = e^x\,dx$, $dv = \cos x$, $v = \sin x$.

$$\int e^x \cos x\ dx = e^x \sin x - \int e^x \sin x\,dx.$$

We apply integration by parts on the second integral: $u = e^x$, $du = e^x\,dx$, $dv = \sin x$, $v = -\cos x$.

$$\int e^x \cos x\ dx = e^x \sin x - (-e^x \cos x + \int e^x \cos x\,dx) = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.$$

Thus,

$$2\int e^x \cos x\ dx = e^x \sin x + e^x \cos x + C,$$

or

$$\int e^x \cos x\ dx = \frac{1}{2}e^x \sin x + \frac{1}{2}e^x \cos x + C.$$

14. (8 points) Evaluate $\int \sec x \tan^3 x\ dx$.

Solution:

$$\int \sec x \tan^3 x\ dx = \int \tan^2 x \sec x \tan x\,dx = \int (\sec^2 x - 1)\sec x \tan x\,dx.$$

Use $u = \sec x$, $du = \sec x \tan x\,dx$:

$$\int \sec x \tan^3 x\ dx = \int u^2 - 1\,du = \frac{1}{3}u^3 - u + C = \frac{1}{3}\sec^3 x - \sec x + C.$$